Test 4
Applications of Integration and Discrete Random Variables

[This test contributes 6% towards the final year mark]

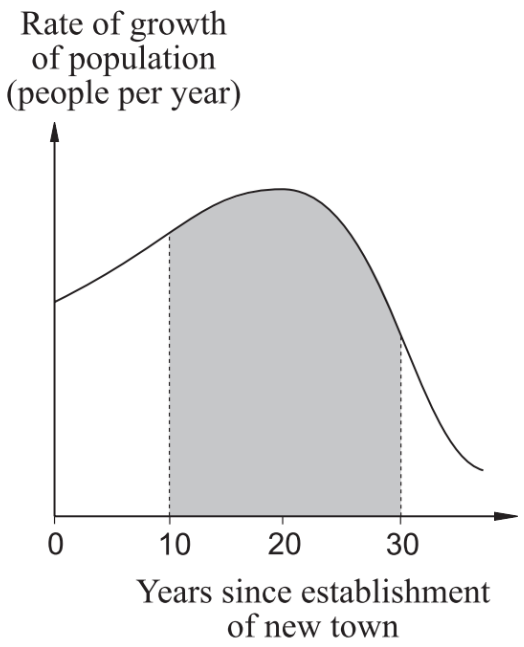
Name : M. Key

Score : (out of 42)

- 40 minutes are allocated for this task.
- CAS and/or scientific calculators are permitted.
- No notes of ANY nature are permitted.
- **Full marks may not be awarded to correct answers unless sufficient justification is given.**
- **Use the method specified (if any) in the question to show your working (otherwise, no marks awarded)**

Do NOT turn over this page until you are instructed to do so.

1. [2 marks]
Describe, in words, what quantity is represented by the shaded area in the graph below.



✓ change in population

The change in population of the new town between the 10th and 30th years since establishment.

✓ between 10th and 30th years

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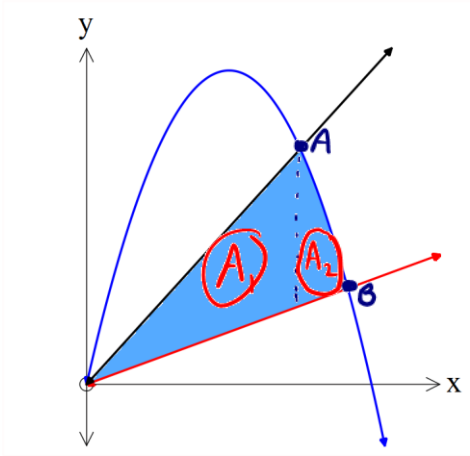
2. [5 marks]
The graph below shows the sketch of the curve $y = 2x(6-x)$ and the lines $y = x$ and $y = 3x$.



Determine the **exact** area of the shaded region.

✓ intersections determined

From CAS, intersection points $A(4.5, 13.5)$ and $B(5.5, 5.5)$

$\therefore$ Shaded area $= \int_0^{4.5} 3x - x \, dx + \int_{4.5}^{5.5} 2x(6-x) - x \, dx$

✓ correct integration limits

$= \int_0^{4.5} 2x \, dx + \int_{4.5}^{5.5} 11x - 2x^2 \, dx$

✓ subtracts x from $3x$ for area A_1

✓ subtracts x from $2x(6-x)$ for area A_2

$= \left[x^2 \right]_0^{4.5} + \left[\frac{11}{2}x^2 - \frac{2}{3}x^3 \right]_{4.5}^{5.5}$

$= (4.5^2 - 0^2) + \left(\frac{11}{2} \times 5.5^2 - \frac{2}{3} \times 5.5^3\right) - \left(\frac{11}{2} \times 5.5^2 - \frac{2}{3} \times 5.5^3\right)$

✓ determines correct area

$= \frac{301}{12}$ sq. units

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3. [3 + 2 + 2 = 7 marks]
A petrol tank, when full, contains 36 litres of petrol. It develops a small hole which widens as time goes by. The rate at which fuel leaks out (in litres per day) is given by the expression:

$$0.009t^2 + 0.08t + 0.01$$

where t is the time in days. When $t = 0$ the tank is full.

(a) Determine a formula for the amount of fuel lost after t days.

✓ variable defined

$\frac{dV}{dt} = 0.009t^2 + 0.08t + 0.01$ where V = amt of fuel lost in L

$\therefore V = \frac{0.009t^3}{3} + \frac{0.08t^2}{2} + 0.01t + c$ ✓ integrated correctly

$\therefore V = 0.003t^3 + 0.04t^2 + 0.01t$ since when $t=0$, $V=0$

✓ constant term considered and determined to be 0

(b) How many litres of fuel does the tank lose on the tenth day?

✓ method

$V(10) - V(9) = 7.1 - 5.517$

$\left(\text{or } \int_9^{10} \frac{dV}{dt}\,dt\right)$ $= 1.583$ L

✓ correct answer

(c) How much fuel is left in the tank after 15 days?

$36 - V(15) = 36 - 19.275$

$= 16.725$ L left

✓ calculates fuel used after 15 days

✓ subtracts from 36 to determine remaining fuel

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4. [4 + 2 + 2 = 8 marks]
An object is thrown vertically upward from a point O (at ground level) with velocity 49 ms^{-1}. The acceleration due to gravity is 9.8 ms^{-2} towards the centre of the Earth.

Determine:

(a) the height above O at any time t,

$a(t) = -9.8$ ✓ correct $a(t)$ given

$\therefore v(t) = \int -9.8\,dt = -9.8t + c$ ✓ constants of integration stated and determined

but $v(0) = 49 \Rightarrow c = 49$

$\therefore v(t) = 49 - 9.8t$ ✓ $v(t)$ obtained

$\therefore x(t) = 49t - 4.9t^2 + k$ but $k = 0$ since $x(0) = 0$

$\therefore x(t) = 49t - 4.9t^2$ ✓ $x(t)$ obtained

(b) the time(s), correct to 3 decimal places, the object is 15 metres above the ground,

i.e. $15 = 49t - 4.9t^2$ ✓ sets up eqn

Solving (CAS) gives $t = 0.316...$ or $t = 9.683...$ ✓ solves for t

i.e. After 0.316 s and 9.684 s (3 d.p.)

(c) the maximum height reached.

max height when $v = 0$

i.e. when $t = \frac{49}{9.8} = 5$ ✓ determines time for max height

$x(5) = 49(5) - 4.9(5)^2$

$= 122.5$ m ✓ determines max height

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5. [3 + 4 + 3 = 10 marks]
The discrete random variable X can only take the values 0, 1, 2, 3, 4, 5. The probability distribution of X is given by the following:

$$P(X=0)=P(X=1)=P(X=2)=a$$
$$P(X=3)=P(X=4)=P(X=5)=b \qquad \text{where } a \text{ and } b \text{ are constants.}$$
$$P(X \geq 2)=3P(X<2)$$

(a) Determine the values of a and b.

$3a + 3b = 1$

$a + 3b = 3(2a)$

Solving (CAS) $a = \frac{1}{8}$, $b = \frac{5}{24}$

✓ uses $\Sigma p = 1$ to determine 1st equation correctly
✓ sets up 2nd equation correctly
✓ solves for a, b correctly

(b) Show that the expectation of X is $\frac{23}{8}$ and determine the ~~exact~~ exact variance of X.

$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{5}{24} + 4 \times \frac{5}{24} + 5 \times \frac{5}{24}$ ✓ uses $E(X) = x.p(x)$

$= \frac{3}{8} + \frac{60}{24}$

$= \frac{23}{8}$

✓ determines $E(X^2)$ correctly ✓ subtracts $(E(X))^2$

$V(X) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{1}{8} + 2^2 \times \frac{1}{8} + 3^2 \times \frac{5}{24} + 4^2 \times \frac{5}{24} + 5^2 \times \frac{5}{24} - \left(\frac{23}{8}\right)^2$

$= \frac{533}{192}$ ✓ determines $V(X)$ correctly

(c) Determine the exact probability that the sum of two independent observations from this distribution exceeds 7.

✓ considers all possible ways

i.e. 5, 5 $\left(\frac{5}{24}\right)^2$

5, 4 (or 4, 5) $2 \times \left(\frac{5}{24}\right)^2$

5, 3 (or 3, 5) $2 \times \left(\frac{5}{24}\right)^2$

4, 4 $\left(\frac{5}{24}\right)^2$

✓ determines probabilities correctly

$\therefore P(\text{sum} > 7) = 6 \times \left(\frac{5}{24}\right)^2 = \frac{150}{576} = \frac{25}{96}$

✓ uses addition rule to calculate correct probability

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6. [3 + 2 + 2 + 3 = 10 marks]
On a long train journey, a statistician is invited by a gambler to play a dice game. The game uses two ordinary dice which the statistician is to throw.

If the total score is 12, the statistician is paid \$6 by the gambler. If the total score is 8, the statistician is paid \$3 by the gambler. However, if both or either dice show a 1, the statistician pays the gambler \$2. Otherwise, no money changes hands.

Let \$ X be the amount paid to the statistician by the gambler.

(a) Complete the table below.

✓ interprets loss as −2

x	-2	0	3	6
$P(X=x)$	$\frac{11}{36}$	$\frac{19}{36}$	$\frac{5}{36}$	$\frac{1}{36}$

✓ subtracts from 1 to determine $P(X=0)$

✓ probs for $x=-2,3,6$ correct

(b) Explain why the table in part (a) describes a probability distribution for the discrete random variable X .

$\Sigma p = 1$ ✓ sum of probs = 1

and $0 \leq p \leq 1$ ✓ each prob between 0 and 1

(c) Show that, if the statistician played the game 100 times, his expected loss would be \$2.78, to the nearest cent.

$$E(X) = \left(-2\times\frac{11}{36}\right) + \left(0\times\frac{15}{36}\right) + \left(3\times\frac{5}{36}\right) + \left(6\times\frac{1}{36}\right)$$

$$= -0.02\dot{7}$$ ✓ determines E(X) correctly

In 100 games he would lose $100\times(-0.02\dot{7}) = -2.\dot{7}$
which is a loss of \$2.78 (2dp) ✓ multiplies E(x) by 100 to get required answer

(d) Find the amount, \$ a , that the \$6 would have to be changed to in order to make the game unbiased.

i.e. $E(X) = 0$ so $\left(-2\times\frac{11}{36}\right) + \left(0\times\frac{15}{36}\right) + \left(3\times\frac{5}{36}\right) + \left(a\times\frac{1}{36}\right) = 0$

✓ recognises $E(X) = 0$

✓ sets up equation correctly

$$-\frac{22}{36} + \frac{15}{36} + \frac{a}{36} = 0$$

$$a = 7$$

✓ solves for a correctly

∴ change to \$7 for a total of 12 for game to be unbiased.

End of test

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